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On the influence of altering the action set on PROMETHEE II's relative ranks

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Abstract

For some multi-criteria decision aiding methods, the relative ranks of two actions may be inverted when the original set is altered. This phenomenon is known as rank reversal. In this contribution, we formalise rank reversal for the PROMETHEE II method and derive the exact conditions for its occurrence when one ore more actions are added or removed from/to the original set. These conditions eventually lead us to: 1) assess whether or not rank reversal between a given pair of actions is, at all, possible, and 2) characterise the evaluations of the actions that have to be added or removed to induce rank reversal. We also propose two metrics that express the "strength" of and the "sensitivity" towards rank reversal; we show, on a toy example, how they could be used in a decision making process.

Keywords: decision making, rank reversal, PROMETHEE II

1 Introduction

Rank reversal (*RR*) is a topic that is addressed in several disciplines, such as social choice theory, economy [13] and psychology [14], decision theory, etc. It has received much attention, because it questions the possible (bounded) rationality of a decision maker (DM). Rank reversal has been reported and studied in numerous multi-criteria decision aid (MCDA) methodologies [16], amongst others in AHP [1, 2, 8, 12, 16], TOPSIS [7], the cross-efficiency evaluation method in data envelopment analysis (DEA), ELECTRE [5, 17], and PROMETHEE [9, 10, 15].

Although often considered as a phenomenon that should be avoided, it has been argued that rank reversal could be legitimate and even desirable [11, 12]. One way of overcoming this debate is to distinguish between the classes of *open* and *closed system* approaches [6, p. 154]. In a nutshell, adding or removing an action from the considered action set in an *open* system will affect the range of scores for the set of actions. For a *closed* system on the contrary, altering the number of actions will lead to the redistribution of action scores in order not to affect the sum of all scores. As PROMETHEE II is a closed system, rank reversal can be observed and should be considered as intrinsically acceptable. However, our goal is not to judge whether rank reversal is legitimate or not, but rather to accept that this phenomenon may occur with PROMETHEE II and to try getting the best possible grip on it. Practically, the added value of our contribution is to give a quantitative and sound bound on when rank reversal may occur when using PROMETHEE II . We also propose several metrics aimed at the practitioner.

For this outranking method in particular, an upper bound for net flow differences between a pair of actions has been proposed above which rank reversal cannot occur [9]. However, despite its simplicity, experiments [10] show that this bound is very pessimistic, thereby reducing its practical applicability. In the present paper, we compute the analytical bound for any pair of

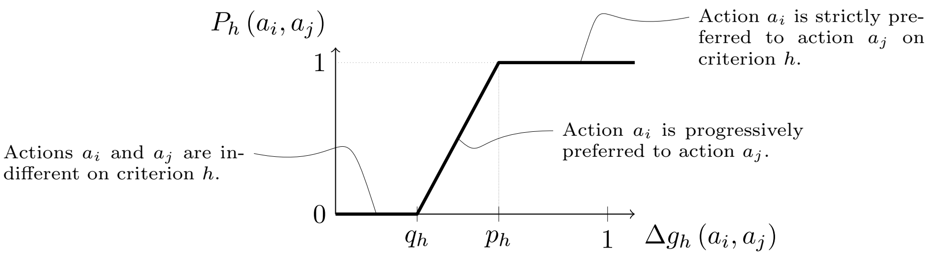


Figure 1: PROMETHEE's "*V-shaped with indifference*" preference function $P_h(a_i, a_j)$ for criterion h, depending on the evaluation difference $\Delta g_h(a_i, a_j)$ of the action pair (a_i, a_j). It is characterised by an indifference threshold q_h and a preference threshold p_h.

actions net flow difference above which rank reversal is not possible. For differences below that bound, rank reversal can always be induced and we provide a characterisation of the actions that lead to it. Based on the preceding, we propose a rank reversal sensitivity measure: the higher this sensitivity, the larger the domain of action evaluations that will lead to rank reversal.

The sequel of the paper is organised as follows: We start by briefly introducing the PROMETHEE II method (Section 2). We then delve into rank reversal in two phases: first, we consider the addition of one single action to the original set and we study its impact with respect to rank reversal on a given pair of actions (Section 3). We then show how this result can be extended to the cases of several added actions, but also to the removal or change of existing actions (Section 4). These theoretical results lay the ground for defining two related metrics that should help the practitioner get a better grip on that phenomenon (Section 5). Finally, we provide an illustrative example to show how rank reversal could be integrated to the analysis of a concrete decision making problem based on PROMETHEE II (Section 6).

2 The PROMETHEE II method

Let $A = \{a_1, \ldots, a_n\}$ be a finite set of n actions. Each action a_i, with $i \in I = \{1, \ldots, n\}$, is characterised by means of an evaluation vector $G(a_i) = \{g_1(a_i), \ldots, g_m(a_i)\} \in \mathcal{G}$, where $g_h(a_i)$ is the evaluation of a_i on criterion $h \in H = \{1, \ldots, m\}$, and $\mathcal{G}$ represents the domain of all possible evaluation vectors. In the following, we assume that this domain is the m-dimensional unit hypercube: $\mathcal{G} = [0, 1]^m$.

To compare any pair of actions $(a_i, a_j) \in A \times A$, a so-called preference function P_h is introduced for each criterion h. It expresses the preference degree, on that criterion, of the first action over the second. Although our approach is not bounded to it, we will consider the widely used "*V-shaped with indifference*" preference function [3] in the following. It uses an indifference and a preference threshold that are respectively denoted by q_h and p_h (Figure 1):

$$P_h(a_i, a_j) = \begin{cases} 0 & \text{, if } \Delta g_h(a_i, a_j) \leq q_h \\ \frac{\Delta g_h(a_i, a_j) - q_h}{p_h - q_h} & \text{, if } q_h < \Delta g_h(a_i, a_j) \leq p_h \\ 1 & \text{, if } p_h < \Delta g_h(a_i, a_j) \end{cases} \quad (1)$$

where $\Delta g_h(a_i, a_j) = g_h(a_i) - g_h(a_j)$ when the criterion has to be maximised, and $\Delta g_h(a_i, a_j) = g_h(a_j) - g_h(a_i)$ otherwise. The pairwise action comparisons are aggregated for each action and provide the unicriterion net flow score $\phi_h(a_i)$ on criterion h:

$$\phi_h(a_i) = \tfrac{1}{n-1} \sum_{j \in I} \Delta P_h(a_i, a_j), \quad (2)$$

where $\Delta P_h(a_i, a_j) = P_h(a_i, a_j) - P_h(a_j, a_i)$. Finally, the unicriterion scores are aggregated once more over all criteria through a weighted sum to yield that action's net flow score:

$$\phi(a_i) = \sum_{h \in H} w_h \phi_h(a_i), \tag{3}$$

where $\{w_1, \ldots, w_m\}$ represents the set of the criteria's relative importance, with $w_h \geq 0, \forall h \in H$, and $\sum_{h \in H} w_h = 1$. For a deeper introduction to the PROMETHEE methods, we refer the reader to [3].

3 Condition for rank reversal

There are two main types of changes in a decision making model that may lead to rank reversal: *i)* changes of the considered actions and/or their evaluations or, *ii)* changes in the preference parameters. In this contribution, we will only address the former ones.

Let us consider an added action x and let $A' = A \cup \{x\}$ be the extended set of $n+1$ actions. Using (2), the net flow score of action a_i with respect to A' becomes:

$$\begin{aligned} \phi'(a_i) &= \tfrac{1}{n} \sum_{h \in H} w_h \Big[\sum_{j \in I} \Delta P_h(a_i, a_j) + \Delta P_h(a_i, x) \Big] \\ &= \tfrac{1}{n} \Big[(n-1)\, \phi(a_i) + \sum_{h \in H} w_h \Delta P_h(a_i, x) \Big]. \end{aligned} \tag{4}$$

The relative rank of two actions, a_i and a_j, is given by the sign of their respective net flows, i.e., a_i is better ranked than a_j if $\Delta\phi(a_i, a_j) = \phi(a_i) - \phi(a_j) > 0$ and reciprocally. Therefore, rank reversal (RR) under the addition of one action x can be expressed as follows:

$$RR(a_i, a_j; x) \quad \Longleftrightarrow \quad \Delta\phi(a_i, a_j)\, \Delta\phi'(a_i, a_j) < 0.$$

Let us make the unrestrictive assumption that a_i is initially better ranked than a_j. This implies that $\Delta\phi(a_i, a_j) > 0$, and therefore simplifies the preceding condition to find the evaluations of x for which $\Delta\phi'(a_i, a_j) < 0$. Integrating (4), the *extended net flow difference* can be rewritten as

$$\Delta\phi'(a_i, a_j) = \tfrac{1}{n} \Big[(n-1)\, \Delta\phi(a_i, a_j) + \sum_{h \in H} w_h z_h(a_i, a_j; x) \Big],$$

where $z_h(a_i, a_j; x) = \Delta P_h(a_i, x) - \Delta P_h(a_j, x)$. Hence, we can express the **exact condition for rank reversal**:

$$RR(a_i, a_j; x) \quad \Longleftrightarrow \quad \sum_{h \in H} w_h z_h(a_i, a_j, x) < -(n-1)\, \Delta\phi(a_i, a_j). \tag{5}$$

Note that, so far, we have not made any assumption regarding the chosen preference function. To be more concrete, we will consider the particular case of the widely used "*V-shaped with indifference*" preference function (Figure 1) in the sequel. Results for other preference function types may, however, be derived in a similar fashion.

Some observations

- The inequality in (5) may be interpreted as follows: An additional action x provokes rank reversal of a pair of actions if the weighted sum of the associated z_h functions is lower than a bound that depends on the number of actions and the initial net flow difference between a_i and a_j. The expression also suggests that a net flow difference between two actions may be considered as some kind of *resistance* to their rank reversal: the higher the net flow difference, the more unlikely their rank reversal. Beyond a certain bound, rank reversal becomes impossible.

- According to (5), each criterion h contributes (with its own weight w_h) to the value of $z(a_i, a_j; x) = \sum_{h \in H} w_h \, z_h(a_i, a_j; x)$. Since we assume that, initially, $\Delta\phi(a_i, a_j) > 0$, the right hand term of (5) will be negative, and, thus, the value of the weighted sum $z(a_i, a_j; x)$ will have to be negative too to induce rank reversal. Looking at the table that describes the possible configurations (Appendix A), we notice that for criterion h, $z_h(a_i, a_j; x)$ can be negative only if the evaluation difference $\Delta g_h(a_i, a_j)$ (which is given by the evaluations of the original set of actions) is negative too. Consequently, for a given pair of actions (a_i, a_j) and assuming a maximisation problem, only the criteria with a negative evaluation difference, i.e., $\Delta g(a_i, a_j) < 0$, are capable of contributing to induce rank reversal. On the contrary, criteria for which $\Delta g(a_i, a_j) > 0$ will tend to reinforce the relative ranking of a_i and a_j. These latter criteria can only, in the best case, not increase the value of $z(a_i, a_j; x)$.
- As has already been pointed out [9], Rank reversal of actions a_i and a_j can only occur if they are mutually non-dominating: there must be at least one criterion for which the unicriterion net flow score difference should be negative, despite a globally positive net flow score difference.

4 Extension to the addition and removal of several actions

Let us generalise the results of the preceding section to the addition of s actions: $S = \{x_1, \ldots, x_s\} \subset \mathcal{A}$ and the removal of r actions: $R = \{y_1, \ldots, y_r\} \subset A$. Note that, while the actions of the first set belong to the set of all possible actions $\mathcal{A}$, the second set, R, only contains actions from the original set A.

Similarly to the case of one additional action x, rank reversal occurs between a_i and a_j if their relative rank before and after having altered the set of actions is reversed:

$$RR(a_i, a_j; S, R) \iff \Delta\phi(a_i, a_j)\,\Delta\phi'(a_i, a_j) < 0.$$

where the altered action set is now defined as $A' = (A \cup S) \setminus R$. Again, assuming that action a_i is initially better ranked than a_j, i.e., that $\Delta\phi(a_i, a_j) > 0$, we obtain the general condition for rank reversal:

$$RR(a_i, a_j; S, R) \iff \sum_{h \in H} w_h \, \mathbf{z}_h(a_i, a_j; S, R) < -(n-1)\,\Delta\phi(a_i, a_j),$$

where the unicriterion z-function is extended to

$$\mathbf{z}_h(a_i, a_j; S, R) = \sum_{x \in S} z_h(a_i, a_j; x) - \sum_{y \in R} z_h(a_i, a_j; y).$$

Again, this condition is independent from the preference function type that is used for each criterion. Note that, as the order of adding and/or removing actions does not matter, changing the evaluations of an action from the set can be conveniently modelled by removing that action and add a new one, that is a modified "copy" of it. We will thus not explicitly consider the change of actions in the sequel.

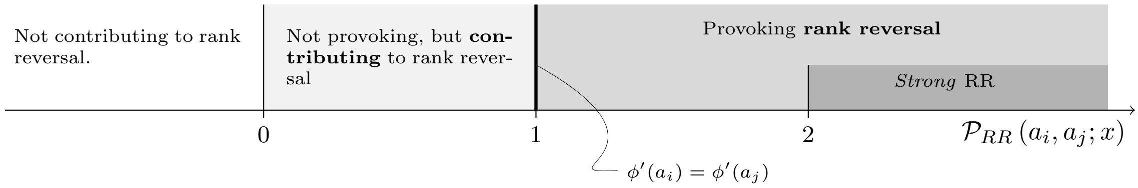


Figure 2: Value domains of the *rank reversing power* $\mathcal{P}_{RR}(a_i, a_j; x)$ and their respective effect on rank reversal of actions a_i and a_j.

5 Some metrics for the practitioner

In this section we propose some metrics that might be useful to quantitatively evaluate rank reversal related characteristics for a particular decision problem. The illustrative example of Section 6 will show how these can be applied in practice.

5.1 How "strongly" are the relative ranks of two actions reversed?

Depending on the evaluations of x, changing the sign of the net flow difference may not be very expressive. Indeed, still assuming that $\Delta\phi(a_i, a_j) > 0$, the above condition would state that there is a rank reversal of two actions, $RR(a_i, a_j)$, even for the smallest negative value of $\Delta\phi'(a_i, a_j)$. It therefore seems sensible to add a measure that expresses the *rank reversing power* of the added action x:

$$P_{RR}(a_i, a_j; x) = \frac{\Delta\phi(a_i, a_j) - \Delta\phi'(a_i, a_j)}{\Delta\phi(a_i, a_j)}. \tag{6}$$

Expressing it in relative terms with respect to the initial net flow difference allows it to be interpreted as follows (Figure 2):

$\mathcal{P}_{RR}(a_i, a_j; x)$	Effect of action x with respect to rank reversal
< 0	x does *not* contribute to provoking rank reversal
$\in]0, 1[$	x contributes but is not sufficient to provoke rank reversal
1	provokes the net flows of a_i and a_j to be equal ($\Delta\phi'(a_i, a_j) = 0$)
$\in]1, 2[$	x provokes rank reversal of a_i and a_j
> 2	x provokes rank reversal "as strong" as a_i and a_j's initial net flow difference

Note that, in the interval $\mathcal{P}_{RR}(a_i, a_j; x) \in [0, 1]$, other "similar" actions need to be added or removed from the set to provoke rank reversal.

If we compute the average of this power on all actions for the second action parameter, we may express the impact of the added action x on action a_i as follows:

$$P_{RR}(a_i; x) = \frac{1}{n} \sum_{j \in I} \max\{0, P_{RR}(a_i, a_j; x)\}$$

For the derived definition, we only consider the positive values of $P_{RR}(a_i, a_j; x)$ in the aggregation process. The aim is to avoid the compensating effect of action pairs for which adding x would actually reinforce their initial relative ranks.

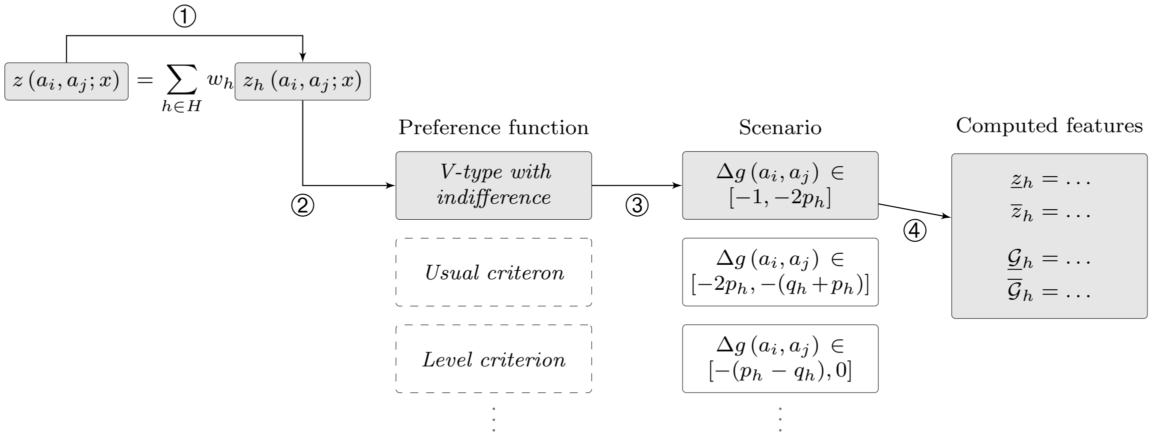


Figure 3: Flowchart of the presented approach. ① We decompose the process by considering each criterion h individually; ② we choose a concrete preference function type to develop the general rank reversal expressions; ③ one specific scenario has to be chosen, depending on the evaluation difference $\Delta g(a_i, a_j)$ and the preference parameters; ④ the actual computation of the characteristic rank reversal features is carried out (or retrieved from Appendix A).

5.2 How sensitive are the ranks of actions?

For the second metric, we take the point of view of the evaluations. Being able to express the likelihood, for a given pair of actions (a_i, a_j) to experience RR is also of practical interest. We therefore propose the following metric:

$$\rho_{RR}(a_i, a_j) = \int_{x \in \mathcal{A}} \delta_{\Delta\phi'(a_i, a_j) < 0} \, \mathrm{d}x, \tag{7}$$

where δ_C evaluates to 1 if the condition C is true and to 0 otherwise; $\mathcal{A}$ represents the set of all possible actions. The range of possible values is $\rho_{RR} \in [0, 1]$. Stated less formally: the more evaluations of x that provoke it, the more the pair (a_i, a_j) is sensitive towards rank reversal. If we assume a uniform distribution of evaluations on all criteria, this value can be interpreted as the probability to see actions a_i and a_j have their rank reversed when an action is added to the original set. However, this interpretation depends on the hypothesis of independence between criteria, which is only rarely met in MCDA problems.

In the next section, we will see on a concrete example that, practically, the integration of (7) can be carried out numerically without difficulty.

6 Illustrative example

The conditions for rank reversal described in the previous sections are very general. We here provide a toy example that illustrates how rank reversal can be "managed" for a concrete data set. Therefor, we will use the following general workflow (Figure 3):

1. The z-function is a weighted sum of unicriterion functions: we consider each criterion individually.

2. For each criterion, a type of preference function is chosen by the decision maker. In this contribution, we only consider the "*V-type with indifference*" function.

3. Having chosen a pair of actions (a_i, a_j) for which we want to test their "rank reversibility", a so-called scenario has to be chosen (Appendix A, left-most column), corresponding to their evaluation difference on that criterion: $\Delta g_h(a_i, a_j) = g_h(a_i) - g_h(a_j)$.

Table 1: Evaluations and preference parameters for our toy example.

Criterion h	Action evaluations $g_h(a_i)$					Pref. parameters		
	a_1	a_2	a_3	a_4	a_5	w_h	q_h	p_h
1	0.75	0.07	0.62	0.57	0.78	0.30	0.05	0.20
2	0.75	0.91	0.40	0.04	0.96	0.70	0.05	0.20

Table 2: Net flows for each action, and net flow differences of all pairs of actions.

Sorted net flows		Net flow differences $\Delta\phi(a_i, a_j)$					
Action	$\phi(a_i)$		a_1	a_2	a_3	a_4	a_5
a_5	0.73	a_1	–	0.05	0.60	0.99	-0.50
a_1	0.23	a_2	-0.05	–	0.55	0.94	-0.55
a_2	0.18	a_3	-0.60	-0.55	–	0.39	-1.10
a_3	-0.37	a_4	-0.99	-0.94	-0.39	–	-1.49
a_4	-0.77	a_5	0.50	0.55	1.10	1.49	–

4. Using Appendix A, the characteristic unicriterion features with respect to rank reversal are deduced for the pair of actions (a_i, a_j).

This procedure is repeated for each criterion. Once aggregated on all criteria, we can compute the global z-function and finally determine whether or not a given additional (or removed) action actually leads to rank reversal.

To be concrete, we consider a randomly generated set of actions (using a uniform distribution on all criteria) and a set of preference parameters (Table 1). For the sake of representability, we limit this example to a decision problem with two criteria to which one single action, x, is added. The approach, however, remains the same for higher dimensions ($m > 2$) and for several actions added and/or removed.

Let us explicitly treat the case of the pair of actions (a_1, a_2): The initial net flow score difference $\Delta\phi(a_1, a_2) = 0.05$ (Table 2) is positive, i.e. action a_1 is initially better ranked than a_2. The evaluation difference on criterion 1 and 2 are respectively: $\Delta g_1(a_1, a_2) = 0.68$ and $\Delta g_2(a_1, a_2) = -0.16$. The latter being negative, it is on criterion 2 that x will contribute to rank reversal, while on criterion 1, it would potentially reinforce the initial relative ranks. Having $q_h = 0.05$ and $p_h = 0.20$ on both criteria, we may deduce the scenario to apply on each. For criterion 1, $\Delta g_1(a_1, a_2) = 0.68 \in [0.40, 1.00] = [2p_h, 1]$: we take the last scenario of Appendix A that yields the minimal value of $z_1(a_1, a_2, x)$: $\underline{z}_1(a_1, a_2) = \min\{z_1(a_1, a_2; 0); z_1(a_1, a_2; 1)\} = \min\{1; 0\} = 0$ (Figure 4).
Likewise for criterion 2, $\Delta g_2(a_1, a_2) = -0.16 \in [-0.25, -0.10] = [-(q_2 + p_2), -2q_h]$, and we take the third scenario to find that $\underline{z}_2(a_1, a_2) = -1$ (Figure 5).

Slightly modifying condition (5), we can determine whether or not actions a_1 and a_2 may, at all, experience rank reversal. Rather than exploring all possible evaluations of the added action x, we notice that the "strongest" possible rank reversal (whatever the actual value of $P_{RR}(a_i; x)$) can be computed through the minimum of the aggregated $z(a_i, a_j; x)$ function. As it is a weighted sum, it suffices to take, for each criterion h individually, the minimal value

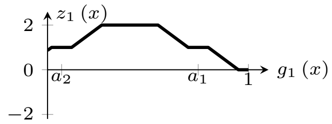


Figure 4: $h = 1$, $\underline{z}_1 = 0$

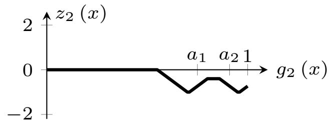


Figure 5: $h = 2$, $\underline{z}_2 = -1$

Table 3: Net flows for each action for the original and the altered set of actions $A' = A \cup \{x\}$.

Original set A		Altered set A'	
Action	$\phi(a_i)$	Action	$\phi'(a_i)$
a_5	0.73	a_5	0.66
		a_6	0.30
$\mathbf{a_1}$	0.22	$\mathbf{a_2}$	0.22
$\mathbf{a_2}$	0.18	$\mathbf{a_1}$	0.12
a_3	-0.37	a_3	-0.50
a_4	-0.77	a_4	-0.81

$\underline{z}_h(a_i, a_j; x)$ that we have already determined:

$$\begin{aligned} \min\{\sum_{h\in H} w_h z_h(a_i, a_j, x)\} = \sum_{h\in H} w_h \underline{z}_h \quad &< \quad -(n-1)\,\Delta\phi(a_i, a_j) \\ 0.30 \times 0 \ + \ 0.70 \times -1 \quad &< \quad -4 \times 0.05 \\ -0.70 \quad &< \quad -0.20 \end{aligned}$$

The condition is satisfied; it is hence possible to find at least one additional action x such that a_1 and a_2 have their relative ranks reversed.

Let us now determine the evaluations of x that lead to the "strongest" rank reversal. For the first criterion, we use Appendix A (last scenario) and Figure 4 to determine that $\underline{\mathcal{G}}_1(a_1, a_2) = [g_1(a_1) + p_1, 1] = [0.95, 1.00]$. Following the same logic (using the third scenario), $\underline{\mathcal{G}}_2(a_1, a_2) = [g_2(a_1) - q_2, g_2(a_2) - p_2] \cup [g_2(a_1) + p_2, g_2(a_2) + q_2] = [0.70, 0.71] \cup [0.95, 0.96]$. The strongest rank reversal will thus occur when $g_1(x) \in [0.95, 1.00]$ and $g_2(x) \in [0.70, 0.71] \cup [0.95, 0.96]$.

As an example, let us choose $g_1(x) = 0.95$ and $g_2(x) = 0.70$. and compute the altered net flow scores (Table 3). For this particular choice, $\Delta\phi'(a_1, a_2) = \phi'(a_1) - \phi'(a_2) = 0.12 - 0.22 = -0.10$. Hence the *rank reversing power* of x with respect to the pair (a_1, a_2) is

$$P_{RR}(a_1, a_2; x) = \frac{\Delta\phi(a_1, a_2) - \Delta\phi'(a_1, a_2)}{\Delta\phi(a_1, a_1)} = \frac{0.05 - (-0.10)}{0.05} = 3$$

According to the definition of $P_{RR}(a_1, a_2; x)$ (Figure 2), the chosen additional action x has a strong rank reversing power. Indeed, $|\Delta\phi'(a_1, a_2)| > |\Delta\phi(a_1, a_2)|$.

Finally, to compute the RR-sensitivity according to (7), we sample the bi-dimensional evaluation space of x. We count the number of sampled evaluations of x that provoke rank reversal of actions a_1 and a_2 and divide the this number by the total number of samples. For a sampling granularity of 0.001 on each criterion, i.e. a total of $1000^2 = 10^6$ samples, we obtain $\rho_{RR}(a_i, a_j) \approx 0.13$. As already mentioned, this could be interpreted as a 13% probability that an action added to the original set provokes a rank reversal of actions a_1 and a_2.

7 Conclusion

Rank reversal is a much debated topic in the MCDA community because it may seem counterintuitive that a third action may alter the relative ranks of two other actions. The PROMETHEE II preference model, as others, is a closed system model and, as such, allows rank reversal. In this paper, we have provided an exact bound that assesses whether a pair of action may experience this phenomenon. A closer look at the evaluation domain of (added or removed) rank reversing third actions shows that the "intensity" of rank reversal may vary. Therefore, we have defined an easy to determine "rank reversing domain *kernel*" $\underline{\mathcal{G}}$. Besides the theoretical interest of having an exact rank reversal bound, there are also several practical advantages:

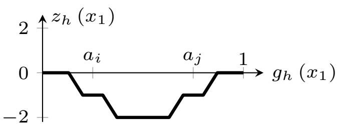
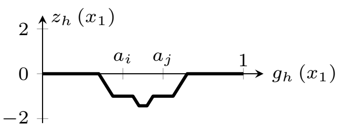
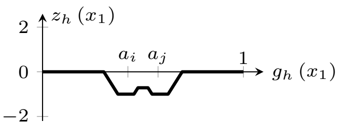
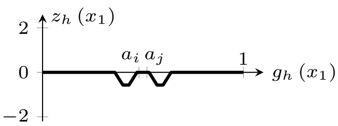
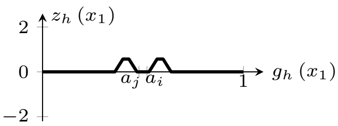
- The stability of one (pair of) action(s) with respect to rank reversal can be evaluated. For a concrete application, this could be integrated to the sensitivity analysis, allowing the decision maker to asses the risk for this phenomenon.
- When increasing the weight of criteria that are less sensitive to rank reversal and decreasing the weights of those that are more sensitive, one could artificially increase the robustness from the point of view of rank reversal. At first sight, this may seem contradictory, because, usually, a decision maker wouldn't want to change her weights to avoid rank reversal. However, one could use this feature for instance in multi-objective optimisation based preference eliciting techniques [4] as an additional objective function: if a set of weight vectors provides the requested ranking, one would probably favour the weight combinations that increase robustness to rank reversal.

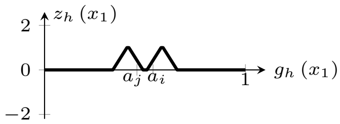
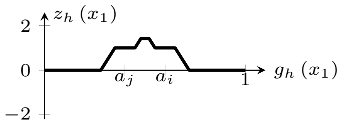
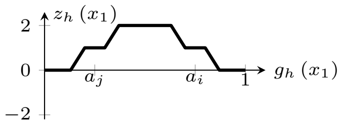
Further investigations could complement the proposed approach. For instance, one could study how the rankings of other actions than the considered pair (a_i, a_j) are affected by a rank reversing action. Would it be meaningful to search for such actions that would exclusively affect the considered pair?

A Domains of extreme values for the $z_h(x)$ function

The following table represents, for each criterion h, the main possible scenarios (other, "degenerated" cases may also happen, but are not represented here for the sake of readability). These depend on the preference parameters, q_h and p_h, as welle as on the evaluation difference $\Delta g_h(a_i, a_j)$. Each row corresponds to one scenario (indicated in the first column):

- The second column schematically depicts the shape of the corresponding z_h function as a function of the evaluation $g_h(x)$ of the added action x on criterion h.
- Finally, the third column indicates the extreme values associated to that scenario:
  - $\underline{z}_h = \min_{g_h(x)\in[0,1]} z_h(a_i, a_j; x)$
  - $\overline{z}_h = \max_{g_h(x)\in[0,1]} z_h(a_i, a_j; x)$
  - $\underline{\mathcal{G}}_h = \arg\min_{g_h(x)\in[0,1]} z_h(a_i, a_j; x)$
  - $\overline{\mathcal{G}}_h = \arg\max_{g_h(x)\in[0,1]} z_h(a_i, a_j; x)$

<table>
<tr><th>$\Delta g_h(a_i, a_j)$</th><th>$z_h(x)$</th><th>Extreme values for action pair (a_i, a_j)</th></tr>
<tr><td>$[-1, -2p_h]$</td><td>
</td><td>$\underline{z}_h = -2$
$\overline{z}_h = \max\{z_h(0); z_h(1)\}$
$\underline{\mathcal{G}}_h = [0,1] \cap [g_{hi} + p_h, g_{hj} - p_h]$
$\overline{\mathcal{G}}_h = [0,1] \setminus [g_{hi} - p_h, g_{hj} + p_h]$</td></tr>
<tr><td>$[-2p_h, -(q_h + p_h)]$</td><td>
</td><td>$\underline{z}_h = -2 + \frac{1}{p_h - q_h}\left(\Delta g_{hij} + 2p_h\right)$
$\overline{z}_h = \max\{z_h(0); z_h(1)\}$
$\underline{\mathcal{G}}_h = [0,1] \cap [g_{hj} - p_h, g_{hi} + p_h]$
$\overline{\mathcal{G}}_h = [0,1] \setminus [g_{hi} - p_h, g_{hj} + p_h]$</td></tr>
<tr><td>$[-(q_h + p_h), -2q_h]$</td><td>
</td><td>$\underline{z}_h = -1$
$\overline{z}_h = \max\left\{z_h(0); z_h\left(\frac{g_{hi}+g_{hj}}{2}\right); z_h(1)\right\}$
$\underline{\mathcal{G}}_h = [0,1] \cap \left([g_{hi} - q_h, g_{hj} - p_h] \cup [g_{hi} + p_h, g_{hj} + q_h]\right)$
$\overline{\mathcal{G}}_h = [0,1] \setminus [g_{hi} - p_h, g_{hj} + p_h]$</td></tr>
<tr><td>$[-2q_h, 0]$</td><td>
</td><td>$\underline{z}_h = -1 + \frac{\Delta g_{hij} + 2(q_h - p_h)}{p_h - q_h}$
$\overline{z}_h = \max\left\{z_h(0); z_h\left(\frac{g_{hi}+g_{hj}}{2}\right); z_h(1)\right\}$
$\underline{\mathcal{G}}_h = [0,1] \cap \left([g_{hj} - q_h, g_{hi} - p_h] \cup [g_{hj} + p_h, g_{hi} + q_h]\right)$
$\overline{\mathcal{G}}_h = [0,1] \setminus \left([g_{hj} - p_h, g_{hi} - q_h] \cup [g_{hj} + q_h, g_{hi} + p_h]\right)$</td></tr>
<tr><td>$[0, p_h - q_h]$</td><td>
</td><td>$\underline{z}_h = \min\left\{z_h(0); z_h\left(\frac{g_{hi}+g_{hj}}{2}\right); z_h(1)\right\}$
$\overline{z}_h = 1 - \frac{\Delta g_{hij} + 2(q_h - p_h)}{p_h - q_h}$
$\underline{\mathcal{G}}_h = [0,1] \setminus \left([g_{hj} - p_h, g_{hi} - q_h] \cup [g_{hj} + q_h, g_{hi} + p_h]\right)$
$\overline{\mathcal{G}}_h = [0,1] \cap \left([g_{hj} - q_h, g_{hi} - p_h] \cup [g_{hj} + p_h, g_{hi} + q_h]\right)$</td></tr>
</table>

$\Delta g_h(a_i, a_j)$	$z_h(x)$	Extreme values for action pair (a_i, a_j)
$[p_h - q_h, q_h + p_h]$		$\underline{z}_h = \min\{z_h(0); z_h\left(\frac{g_{hi}+g_{hj}}{2}\right); z_h(1)\}$ $\overline{z}_h = 1$ $\underline{\mathcal{G}}_h = [0,1] \setminus [g_{hj} - p_h, g_{hi} + p_h]$ $\overline{\mathcal{G}}_h = [0,1] \cap ([g_{hj} - q_h, g_{hi} - p_h] \cup [g_{hj} + p_h, g_{hi} + q_h])$
$[q_h + p_h, 2p_h]$		$\underline{z}_h = \min\{z_h(0); z_h(1)\}$ $\overline{z}_h = 2 - \frac{1}{p_h - q_h}(\Delta g_{hij} + 2p_h)$ $\underline{\mathcal{G}}_h = [0,1] \setminus [g_{hj} - p_h, g_{hi} + p_h]$ $\overline{\mathcal{G}}_h = [0,1] \cap [g_{hi} - p_h, g_{hj} + p_h]$
$[2p_h, 1]$		$\underline{z}_h = \min\{z_h(0); z_h(1)\}$ $\overline{z}_h = 2$ $\underline{\mathcal{G}}_h = [0,1] \setminus [g_{hj} - p_h, g_{hi} + p_h]$ $\overline{\mathcal{G}}_h = [0,1] \cap [g_{hj} + p_h, g_{hi} - p_h]$

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