SMAA-GAIA: A Complementary Tool of the SMAA-PROMETHEE Method

CoDE-SMG – Technical Report Series

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CoDE-SMG – Technical Report Series

TR/SMG/2014-005

August 2014
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ABSTRACT

PROMETHEE and GAIA are well-known Multiple Criteria Decision Aid methods. Given an evaluation table and preference parameters they allow to rank the alternatives, to visualize the problem, to perform sensitivity and robustness analysis, etc. Unfortunately, it is often hard for the Decision Maker (DM) to estimate the precise values of these parameters. Therefore an alternative option is to give ranges of potential values in order to apply Stochastic Multicriteria Acceptability Analysis. This has been recently studied in the context of the SMAA-PROMETHEE method. The aim of this contribution is to propose a SMAA extension of GAIA. We show how this tool can be useful and provide complementary information to SMAA-PROMETHEE. This is illustrated on a pedagogical example.

Keywords - MCDA, PROMETHEE, GAIA, SMAA

1 INTRODUCTION

PROMETHEE[5] is a Multiple Criteria Decision Aid method (MCDA) that provides a complete or partial ranking of different alternatives based on net flow scores. GAIA[3] is a complementary tool to visualize the multicriteria problem. These approaches have been used in hundreds of applications including healthcare, finance, logistics, water management, etc[1]. In order to apply the PROMETHEE methodology, the Decision Maker (DM) usually provides an evaluation table but also preferential information such as weights, indifference and preference thresholds. However, the conclusions obtained with PROMETHEE or GAIA may sometimes be sensitive to these values. It is thus important to investigate the robustness of the results. On the one hand, the decision maker can perform an ex-post analysis with tools such as walking weights, stability intervals, decision maker brain[4]. On the other hand, he/she can specify some possible stochastic distribution of the input parameters. This situation has recently been investigated by applying Stochastic Multicriteria Acceptability Analysis (SMAA) in the context of PROMETHEE[6]. In this contribution, we further complete this proposal by extending GAIA with SMAA. We will briefly present PROMETHEE and GAIA. Then we will show through an example how SMAA can be applied and useful to GAIA.

2 PROMETHEE II AND GAIA

Let us consider a set of alternatives denoted \( A = \{a_1, \ldots, a_n\} \) evaluated on a family of criteria denoted \( C = \{c_1, \ldots, c_k\} \) where \( c_j : A \rightarrow \mathbb{R} \), one
builds a preference function \( P_j(a, b) \) comparing each pair of alternatives on every criterion. This function is based on the difference \( d_j(a, b) = c_j(a) - c_j(b) \) between two evaluations[2]. Usually six standard functions are proposed to the DM. For the sake of simplicity we only consider a linear function with an indifference and a preference threshold respectively denoted \( q_j \) and \( p_j \):

\[
P_j(a, b) = \begin{cases} 
0 & \text{if } d_j(a, b) \leq q_j, \\
\frac{d_j(a, b) - q_j}{p_j - q_j}, & \text{if } q_j < d_j(a, b) < p_j, \\
1 & \text{if } d_j(a, b) \geq p_j.
\end{cases}
\]

Knowing the weight \( w_j \) of each criterion, one computes

\[
\pi(a, b) = \sum_{j=1}^{k} w_j P_j(a, b), 
\]

which quantifies how much is the alternative \( a \) preferred to \( b \), under the conditions that \( \sum_{j=1}^{k} w_j = 1 \), \( w_j \geq 0 \).

One defines the positive and the negative outranking flows of an alternative respectively denoted \( \phi^+(a) \) and \( \phi^-(b) \):

\[
\phi^+(a) = \frac{1}{n-1} \sum_{i=1}^{n} \pi(a, a_i) \quad (2)
\]

\[
\phi^-(a) = \frac{1}{n-1} \sum_{i=1}^{n} \pi(a_i, a) \quad (3)
\]

Finally, one computes the net flow of an alternative \( \phi(a) \),

\[
\phi(a) = \phi^+(a) - \phi^-(a) \quad (4)
\]

A complete PROMETHEE II ranking can be built following the next rule[5]:

\[
\begin{cases} 
ap^{11}b & \text{if } \phi(a) > \phi(b), 
nap^{11}b & \text{if } \phi(a) = \phi(b). 
\end{cases}
\]

According to (1), (2) and (3), we have

\[
\phi(a) = \frac{1}{n-1} \sum_{j=1}^{k} \sum_{i=1}^{n} [P_j(a, a_i) - P_j(a_i, a)], w_j 
\]

Consequently,

\[
\phi(a) = \sum_{j=1}^{k} \phi_j(a), w_j 
\]

Where

\[
\phi_j(a) = \frac{1}{n-1} \sum_{i=1}^{n} [P_j(a, a_i) - P_j(a_i, a)] \quad (7)
\]

\( \phi_j \) is defined as the unicriterion net flow score. Let \( \Phi \) define the matrix of the unicriterion flow scores for each criterion and each alternative:

\[
\Phi = \begin{bmatrix} 
\phi_1(a_1) & \cdots & \phi_k(a_1) \\
\vdots & \ddots & \vdots \\
\phi_1(a_n) & \cdots & \phi_k(a_n) 
\end{bmatrix} \quad (8)
\]

One can build a variance/covariance matrix of \( \Phi[2] \),

\[
\Phi_{cov} = \frac{1}{n} \Phi' \Phi \quad (9)
\]

GAIA is a complementary visual tool based on a PCA applied to the unicriterion net flows space. The underlying idea is to represent the multicriteria problem on a plan. Therefore one finds the two largest eigenvalues \( \lambda_1, \lambda_2 \) and the related eigenvectors to define this projection plan. The loss of information can be computed as

\[
1 - \delta = \frac{1 - (\lambda_1 + \lambda_2)}{\sum_{j=1}^{k} \lambda_j} \quad (10)
\]

We refer the interested reader to [9] for a detailed description of GAIA.

2.1 SMAA-GAIA

Let us first remind the reader that the extension of PROMETHEE to SMAA has recently been investigated by [6]. We refer the interested reader to this contribution for a detailed discussion. As already stressed, the different evaluations that are required from the DM are sometimes difficult to determine precisely. First because the evaluations of the criteria might be estimations and because the DM may hesitate on the choice of crisp weights and thresholds values. In order to address these issues one may model these values as random variables and apply the SMAA method. To keep it simple we will assume that the DM provides a range of possible values for the evaluations of the criteria, the thresholds and the weights. This will be determined by a lower bound, a higher bound and in between, the most probable value. This is modeled by a triangular distribution as represented on figure 1. The lower
bound is \( \alpha \), the more probable value is \( \gamma \) and the higher bound is \( \beta \).

The DM might be interested in knowing the alternative that came out first with the largest probability. He may also want to know what range of ranks all the alternatives may have, etc.

**Figure 1: Triangular density function**

\[
\begin{align*}
\gamma_i &:= \frac{2}{(\beta_i - \alpha_i)} \\
\alpha_i &:= \text{lower bound} \\
\beta_i &:= \text{higher bound} \\
C_j(a_i) &:= \text{weight}
\end{align*}
\]

SMAA-GAIA is a solution to represent this range of results on the GAIA plan. A way to show it is to compute a bivariate kernel density estimation [7] of all the stochastic net flows for each alternative. The GAIA plan is built in a traditional way by only considering the more probable values of the performance table. The kernel density estimation is obtained by dividing the GAIA plan into a mesh. Then one estimates the proportions of projections around each noodle of the mesh by using the Parzen method [10]. It builds a gaussian kernel around the noodles. The result is the superposition of all the gaussian kernels. Finally, it is projected on the GAIA plan. It provides a useful visualization of the behavior of each alternative under stochastic constraints. Moreover, instead of projecting all the weights sampled, only 95% of them are represented by drawing the convex hull of these points. So the the extreme weights are eliminated and the DM only visualize the problem with the more probable values. The 95% are chosen by considering the nearest from the mean value of all the weight. Of course, this parameter can be set by the DM to higher or lower values.

Up to this point, we have assumed that random evaluations of the alternatives are independent. Yet, most of the time the criteria are correlated. As example, if the price of a car decreases, then its power is likely to decrease too. In order to take into account this distinctive feature, we propose to consider the correlation matrix. Either the DM is able to provide such values or in regards of the performance table, a covariance matrix \( C \) can be estimated. Thus the problem is to generate \( X = \{X_1 \cdots X_k\} \) where \( X \sim \text{MN}(0, C) \). Let \( Z \) be a random variable such that \( Z \sim \text{N}(0, 1) \). Let \( L \) be the square root of \( C \), i.e, \( C = L L' \) (using the Cholesky decomposition). We have \( X = L' Z \). The new evaluation of \( a_i \) on \( c_j \) is \( \beta_1 + X_i \). However, one has to remind that this matrix is built with the different alternatives. The more alternatives, the more accurate the covariance matrix.

3 RESULTS AND DISCUSSION

In order to illustrate the application of SMAA-GAIA we consider the same example as in [6] (presented on table 1).

<table>
<thead>
<tr>
<th>Car</th>
<th>Net Flow</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>PE</td>
<td>0.1684</td>
<td>1</td>
</tr>
<tr>
<td>CI</td>
<td>-0.0587</td>
<td>4</td>
</tr>
<tr>
<td>FI</td>
<td>-0.0538</td>
<td>3</td>
</tr>
<tr>
<td>SK</td>
<td>0.0314</td>
<td>2</td>
</tr>
<tr>
<td>LA</td>
<td>-0.0873</td>
<td>5</td>
</tr>
</tbody>
</table>

The PROMETHEE II ranking presents the first car (PE) at the first rank and the forth car (SK) at the second rank. Still considering this example, but using a range and a more probable value (instead of precise values), we have the performance table 2. The more probable values are those used to compute the net flow scores in table 3. The SMAA conclusion are presented on tables 4 and 5.

<table>
<thead>
<tr>
<th>Cars</th>
<th>Worst Ranking</th>
<th>Best Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>PE</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>CI</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>FI</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>SK</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>LA</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3: PROMETHEE II, a Classical Result

Table 4: Best and Worst ranking at 95% Chance
Let us stress that the number of different ranking rises to 107. There is thus 89% of the 120 possible rankings appears.

Considering the performance table 2, the figures 2 and 3 show the results of the complementary tool SMAA-GAIA respectively for a uniform distribution as[6] and for a triangular distribution.

In order to obtain these values we simulated 1000 evaluations of the alternatives. The loss of information (in %) caused by the projection equals 1 – δ where δ = 0.9855. What appears instantly is that there is a sensible difference between the two methods. The choice of the random numbers generation method is important.

If the DM can estimate more probable values, he should provide them in order to present a more realistic result. Besides, one easily understands how two alternatives could be considered as equal as their KDE projection are mixed. The less mixed and the less spread, the more robust. Besides, one can understand the impact of possible weight values in the final choice. In our example, one quickly understands that FI and LA will be very sensitive to the values of the performance table because the KDE of LA is included in the KDE of FI whereas SK will be less sensitive to the fluctuations of the values. This effect is even more striking using a triangular distribution because the zone of largest density of evaluations of the alternatives. The the loss

One easily understands that even advantaging the first car (PE), it still has only around 4.9% chance of being the best car (this is the reason why its best position is equal to 2 in table 4). Whereas the forth car (SK) is probably the best choice : its worst rank is the third and its best is the first rank with 80.7% probability. This shows the robustness of the conclusion of PROMETHEE. In this case, thanks to SMAA-PROMETHEE we have seen that the conclusions of the initial PROMETHEE II analysis was not robust. The DM can safely buy the Skoda instead of the Peugeot proposed at first sight.

### Table 1: Performance table: the cars - PROMETHEE II

<table>
<thead>
<tr>
<th>Cars</th>
<th>Price [Euro]</th>
<th>Acceleration from 0 to 100 kilometer/hour</th>
<th>Max speed [km/h]</th>
<th>Consumption [l/100 km]</th>
</tr>
</thead>
<tbody>
<tr>
<td>PE</td>
<td>PEUGEOT 208 1.6 8 V</td>
<td>16295</td>
<td>10.57</td>
<td>183.35</td>
</tr>
<tr>
<td>CI</td>
<td>CITROEN C3</td>
<td>15800</td>
<td>13.39</td>
<td>162.85</td>
</tr>
<tr>
<td>FI</td>
<td>FIAT 500.0 0</td>
<td>15230</td>
<td>11.08</td>
<td>172.53</td>
</tr>
<tr>
<td>SK</td>
<td>SKODA Fabia 1.2</td>
<td>15250</td>
<td>14.09</td>
<td>170.21</td>
</tr>
<tr>
<td>LA</td>
<td>LANCIA Ypsilon 5p</td>
<td>16170</td>
<td>11.77</td>
<td>183.87</td>
</tr>
<tr>
<td>qi</td>
<td>773</td>
<td>2.92</td>
<td>30</td>
<td>0.22</td>
</tr>
<tr>
<td>pi</td>
<td>1778</td>
<td>4.02</td>
<td>40</td>
<td>0.5</td>
</tr>
<tr>
<td>wi</td>
<td>0.52</td>
<td>0.14</td>
<td>0.06</td>
<td>0.27</td>
</tr>
</tbody>
</table>

### Table 2: Performance table: the cars with uncertain evaluations

<table>
<thead>
<tr>
<th>Cars</th>
<th>Price [Euro]</th>
<th>Acceleration from 0 to 100 kilometer/hour</th>
<th>Max speed [km/h]</th>
<th>Consumption [l/100 km]</th>
</tr>
</thead>
<tbody>
<tr>
<td>PE</td>
<td>[16000,16295,18000]</td>
<td>[9.5, 10.57, 11.5]</td>
<td>[175, 183.35, 190]</td>
<td>[2.8, 3.09, 4.5]</td>
</tr>
<tr>
<td>CI</td>
<td>[15000, 15800, 16500]</td>
<td>[12.7, 13.39, 14.2]</td>
<td>[155, 162.85, 170]</td>
<td>[3.4, 3.92, 4.6]</td>
</tr>
<tr>
<td>FI</td>
<td>[14500, 15230, 15800]</td>
<td>[10, 11.08, 12]</td>
<td>[165, 172.53, 180]</td>
<td>[3.4, 4.15, 5]</td>
</tr>
<tr>
<td>SK</td>
<td>[14100, 15250, 15650]</td>
<td>[13.2, 14.09, 15.2]</td>
<td>[160, 170.21, 181]</td>
<td>[2.5, 3.89, 4.3]</td>
</tr>
<tr>
<td>LA</td>
<td>[15000, 16170, 17100]</td>
<td>[10.6, 11.77, 12.8]</td>
<td>[175, 183.87, 191]</td>
<td>[3.2, 3.84, 4.4]</td>
</tr>
<tr>
<td>qi</td>
<td>[500, 773, 1000]</td>
<td>[2, 2.52, 3]</td>
<td>[30, 30, 30]</td>
<td>[0, 0.22, 0.5]</td>
</tr>
<tr>
<td>pi</td>
<td>[1500, 1778, 2000]</td>
<td>[3, 4.02, 5]</td>
<td>[40, 40, 40]</td>
<td>[0.5, 0.5, 0.5]</td>
</tr>
<tr>
<td>wi</td>
<td>[0, 0.52, 1]</td>
<td>[0, 0.14, 1]</td>
<td>[0, 0.06, 1]</td>
<td>[0, 0.27, 1]</td>
</tr>
</tbody>
</table>
The same results (but using a covariance matrix estimated on the most probable evaluations) are presented on figures 4 and 5. As expected, one can see that the results are different than without the covariance matrix. Now the selection of SK is less obvious and PE appears to be another choice. Let us stress that these results have been obtained by using Matlab and associated functions.

4 CONCLUSION

SMAA PROMETHEE has recently been proposed in order to investigate the robustness of multicriteria rankings when input parameters (evaluations and preference values) are subject to random distributions. This method can enforce the DM confidence in his choice or point out sensitive conclusions. The aim of the contribution was to introduce a complementary tool, called SMAA-GAIA, that allows to visualize such stochastic results in a two dimensional plan. This can help the decision maker to identify alternatives that are likely to move in the same zones (or not) when randomness is taken into account. The shape and the distance between these regions give some insights about the robustness of the alternatives (with respect to the different criteria and the decision stick). Finally, we also introduced potential correlations between the random variables in order to simulate more realistic profiles. In order to
compute the GAIA plan, we have decided to base the identification of the eigenvectors on the uniparameter net flow scores computed based on the most probable values. This can be viewed as a limitation (since plenty of GAIA plans could potentially be computed in functions of the different instantiations of the random variables). For instance another option could be to identify both pessimistic and optimistic GAIA plan and try to aggregate such kinds of information. Finally, it seems that preference elicitation of triangular distributions (and potential correlations) could also be an interesting research question.

REFERENCES


