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# The Inverse Multi-Objective $\{0, 1\}$-Knapsack Problem under the Chebyshev Distance

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Julien Roland juroland@ulb.ac.be
Yves De Smet yves.de.smet@ulb.ac.be
José Rui Figueira jose.figueira@mines.inpl-nancy.fr
CoDE-SMG, Université Libre de Bruxelles, Brussels, Belgium

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**Abstract**

The inverse multi-objective {0,1}-knapsack problem consists of finding a minimal adjustment of the profit matrix such that a given feasible set of items becomes an efficient solution. In this paper, the adjustment is measured by the Chebyshev distance. It is shown how to build an optimal adjustement in linear time based on this distance, and why it is right to perform a binary search for determining the optimal distance. These theoretical results led us to propose an algorithm based on the resolution of mixed-integer linear programs.

## 1 Introduction

Inverse optimization is a very recent mathematical programming model with applications in a wide range of area such as geophysical sciences, transportation, and traffic flow (see for example Tarantola, 1987; Burton and Toint, 1992; Sokkalingam et al., 1999; Ahuja and Orlin, 2001).

In real-world applications, it is common to face decision making problems that are by their very nature multidimentional. This is a prohiminent aspect in situations modeled as knapsack problems, such as in capital budgeting (Bhaskar, 1979; Rosenblatt and Sinuany-Stern, 1989), planning remediation of contaminated lightstation sites (Jenkins, 2002), selecting transportation investments (Teng and Tzeng, 1996), and relocating problems arising in conservation biology (Kostreva et al., 1999). In capital budgeting one could maximize the net present value of the selected projects and simulteanously minimize their risk. To our best knowledge, this aspect has not been taken into account in inverse optimization. In this paper, our aim is to go a step further extending inverse optimization to multi-objective optimization problems. This extension has been mentioned in Ahuja and Orlin (2001), but not covered yet.

An illustration of inverse optimization in the field of multi-objective optimization drawn from portfolio analysis can be stated as follows. Portfolios are generally built by searching the ones that minimize risk and simulteanously maximize return, dividends, liquidity, etc. Given a portfolio, it is nevertheless difficult to evaluate its performances on this set of objectives. In such a kind of problems it is common to model the expected return by the average daily return. However, when looking at the period to observe, there is no consensus among researchers and practitioners. It might be calculated over the last week, the last month, etc. One could start to build a general model in order to identify efficient solutions. An observed portfolio might be dominated in this model because investors *percieve* or *evaluate* parameters in a slightly different way. However, a natural constraint is to impose the efficiency of observed portfolios. Therefore, one could focus on the minimal adjustment of the initial model parameters in order to satisfy this constraint.

In a first attempt for solving this problem, it is assumed that all the objectives and constraints are linear. Within these assumptions, the model is an instance of the inverse multi-objective knapsack problem. Given an instance of the knapsack problem and a feasible solution, the question is how to modify the profits as little as possible such that the given solution becomes an efficient one.

The paper deals with this problem where the adjustment of the profits is measured by the Chebyshev distance. Some theoretical results are provided and an algorithm is proposed. This algorithm performs a particular search on the set of profit matrices in order to find the minimal adjustment, which makes the given feasible solution an efficient one. To achieve this purpose, the algorithm makes use of a mixed-integer program in order to test the efficiency of the feasible solution for the given profit matrix.

This paper is organized as follows. In Section 2, concepts, definitions, and notation are introduced. In Section 3, the inverse knapsack problem is formally defined, preliminar results are highlighted, and an algorithm is proposed to solve the problem. We conclude with remarks and avenues for future research.

## 2 Concepts, Definitions, and Notation

This section addresses several concepts, their definitions, and the basic notation used in the remaining sections of this paper.

Let $\mathbb{R}^n = \{(x_1, x_2, \dots, x_j, \dots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, 2, \dots, n\}$ denote the set of real-valued vectors of length $n \geqslant 1$, $\mathbb{R}^{q \times n} = \{(c^1, c^2, \dots, c^i, \dots, c^q) : c^i \in \mathbb{R}^n \text{ for } i = 1, 2, \dots, q\}$ the set of real-valued matrices composed of $n$ columns and $q$ rows denoted by $c^i$ with $i = 1, 2, \dots, q$. A vector $x \in \mathbb{R}^n$ is a matrix composed of 1 column and $n$ rows, and the transpose of $x$, denoted by $x^T$, is a matrix composed of $n$ columns and 1 row. The canonical basis of $\mathbb{R}^n$ is denoted by the $n$ vectors $e^j$ with $j = 1, 2, \dots, n$.

Let $x, y \in R^n$ be two vectors. The following notation will be used hereafter: $x < y$ iff $\forall j \in \{1, 2, \dots, n\} : x_j < y_j$; $x \leqq y$ iff $\forall j \in \{1, 2, \dots, n\} : x_j \leqslant y_j$; $x \neq y$ iff $\exists j \in \{1, 2, \dots, n\} : x_j \neq y_j$; $x \leq y$ iff $x \leqq y$ and $x \neq y$. The binary relations $\geqq$, $\geq$, and $>$ are defined in a similar way.

Let $V, W \subseteq \mathbb{R}^n$ denote two sets of vectors. Then, the set addition of $V$ and $W$ (denoted by $V \oplus W$) can be stated as follows, $V \oplus W = \{x \in \mathbb{R}^n : x = x^1 + x^2, x^1 \in V, x^2 \in W\}$. For the sake of simplicity $x \oplus W$ is used instead of $\{x\} \oplus W$.

A subset $S \subseteq \mathbb{R}^n$ is a cone if and only if $x^1, x^2 \in S$ implies that $\lambda_1 x^1 + \lambda_2 x^2 \in S$, for all $\lambda_1 \geqslant 0$ and $\lambda_2 \geqslant 0$ (Padberg, 1995). Finally, let us define the following two cones:

$$\mathbb{R}^n_{\geqq} = \Big\{x \in \mathbb{R}^n : x = \sum_{i=1}^{n} e^i \alpha_i,\ \alpha_i \geqslant 0,\ i = 1, \dots, n\Big\},$$

$$\mathbb{R}^n_{\leqq} = \Big\{x \in \mathbb{R}^n : x = \sum_{i=1}^{n} e^i \alpha_i,\ \alpha_i \leqslant 0,\ i = 1, \dots, n\Big\}.$$

The Multi-Objective {0,1}-Knapsack problem (MKP) is a well-known classical multi-objective linear {0,1}-combinatorial optimization problem (see, for example, Kellerer et al. (1994)). Let $J$ denote a set composed of $n$ items with $q$ profits $c^i_j \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, for $i \in I$, and weights $w_j \in \mathbb{N}_0$, for each $j \in J$. Let $W \in \mathbb{N}_0$ denote the capacity of the knapsack. The MKP problem consists of selecting a subset $S \subseteq J$, such that the sum of the profits of the elements of $S$ is "maximized" and the sum of weights of the same elements does not exceed the capacity of the knapsack.

The problem can be modeled as a multi-objective binary linear programming problem as follows.

$$\begin{array}{rll} \text{“max”} & F(x) = \{f^1(x), f^2(x), \dots, f^i(x), \dots, f^q(x)\} & \\ \text{subject to:} & \sum_{j \in J} w_j x_j \quad \leqslant W & \qquad \text{(MKP)} \\ & \qquad x_j \quad \in \{0, 1\},\ j \in J. & \end{array}$$

where $\forall i \in I : f^i(x) = \sum_{j\in J} c^i_j x_j$. Consequently, an instance of the MKP problem is defined by a feasible set $X = \{x \in \{0,1\}^n : \sum_{j\in J} w_j x_j \leqslant W\}$, and a profit matrix $C \in \mathbb{N}_0^{q\times n}$. Let (X,C) denote such an instance. It is assumed that $\sum_{j\in J} w_j > W$ and $w_j \leqslant W$, for all $j \in J$, otherwise, the problem is obvious.

In multi-objective optimization two spaces should be distinguished. The *decision space*, *i.e.*, the space in which the feasible solutions are defined, and the *objective space*, *i.e.*, the space in which the outcome vectors are defined. The image of the feasible set in the objective space is denoted by $Y = \{y \in \mathbb{R}^q : y = Cx, x \in X\}$.

An outcome vector $y^* \in Y$ is said to be ideal if and only if $Y \subseteq \{y^* \oplus \mathbb{R}^q_{\leqq}\}$. In the definition of MKP, the maximization operator is between quotation marks since there is not necessarily an ideal outcome vector and that no natural total order exists on $\mathbb{R}^q$ with $q \geqslant 2$. Consequently, it is widely accepted to build the dominance relation on the set $Y$ of the outcome vectors.

**Definition 2.1** (Dominance)**.** *Let $y, y' \in Y$ denote two outcome vectors such that $y \neq y'$. If $y' \in \{y \oplus \mathbb{R}^q_{\leqq}\}$, then $y$ dominates $y'$.*

*Dominance* is a binary relation that is irreflexive, asymmetric, and transtitive. This relation induces a partition of $Y$ into two subsets: the set of dominated outcome vectors and the set of *non-dominated* outcome vectors. An outcome vector $y^* \in Y$ is non-dominated if there is no $y \in Y$ such that $y$ dominates $y^*$. Let $y \in Y$ be an outcome vector. If $\{y \oplus \mathbb{R}^q_{\geqq}\} \cap Y = \{y\}$, then $y$ is non-dominated. Conversely, in the decision space the concepts of *efficient* and *non-efficient* solutions can be defined. A solution $x^* \in X$ is efficient if and only if there is no $x \in X$ such that $Cx \geq Cx^*$.

# 3 Theoretical Developments

The question adressed in this paper is how to determine a minimal adjustment of the profit matrix such that a given feasible solution becomes efficient. Let us start by defining a linear integer program to check wether or not $Dx^0$ is a non-dominated solution of $(X, D)$.

$$\begin{array}{rl} s^* \in \arg\max \sum_{i\in I} s_i & \\ \text{subject to:} & \sum_{j\in J} d^i_j x_j \geqslant \sum_{j\in J} d^i_j x^0_j + s_i,\ i \in I \\ & s_i \geqslant 0,\ i \in I \\ & x \in X \end{array} \tag{1}$$

It is obvious that $\sum_{i\in I} s^*_i = 0$ if and ony if $Dx^0$ is a non-dominated solution of $(X, D)$.

Let $(X, C)$ denote an instance of MKP and $x^0 \in X$ a feasible solution. Consider the $L_\infty$ distance (Chebyshev distance) between two matrices $C$ and $D$, *i.e.*, $\max_{i\in I, j\in J} |c^i_j - d^i_j|$. The $L_\infty$ *inverse multi-objective* $\{0,1\}$*-knapsack problem* (IMKP-$\infty$) can be stated as follows:

$$\begin{array}{rl} D^* \in \arg\min & \max_{i\in I, j\in J} |c^i_j - d^i_j| \\ \text{subject to:} & \sum_{i\in I} s^*_i = 0 \\ & D \in \mathbb{N}_0^{2\times n} \\ & \begin{array}{rl} s^* \in \arg\max \sum_{i\in I} s_i & \\ \text{subject to:} & \sum_{j\in J} d^i_j x_j \geqslant \sum_{j\in J} d^i_j x^0_j + s_i,\ i \in I \\ & s_i \geqslant 0,\ i \in I \\ & x \in X \end{array} \end{array} \tag{IMKP-$\infty$}$$

IMKP-$\infty$ is a bilevel optimization problem were we seek for a profit matrix $D^* \in \mathbb{N}_0^{q\times n}$, which minimizes the $L_\infty$ distance with respect to $C$ and such that $x^0$ is an efficient solution of the modified knapsack problem $(X, D^*)$.

Let us analyze the nature of some optimal solutions of IMKP-$\infty$. Based on a partition of $J$ defined by $J^0 = \{j \in J : x^0_j = 0\}$ and $J^1 = \{j \in J : x^0_j = 1\}$, the first theorem establishes that an optimal solution $D^*$ can be built by increasing $c^i_j$, for all $j \in J^1$ and by decreasing $c^i_j$, for all $j \in J^0$, for $i \in I$.

**Theorem 3.1.** *There exists an optimal solution $D^* \in \mathbb{N}^{q\times n}$ of IMKP-$\infty$ such that $\forall j \in J^1$ : $d_j^{i*} \geqslant c_j^i$ and $\forall j \in J^0 : d_j^{i*} \leqslant c_j^i$, with $i \in I$.*

*Proof.* Let $D \in \mathbb{N}^{q\times n}$ denote any optimal solution of IMKP-$\infty$. Define the following sets for all $i \in I$ : $J_i^{0>} = \{j \in J^0 : d_j^i > c_j^i\}$, $J_i^{0\leqslant} = \{j \in J^0 : d_j^i \leqslant c_j^i\}$, $J_i^{1<} = \{j \in J^1 : d_j^i < c_j^i\}$, $J_i^{1\geqslant} = \{j \in J^1 : d_j^i \geqslant c_j^i\}$. Consider a solution $D^*$ of IMKP-$\infty$ defined as follows, for all $i \in I$:

$$d_j^{i*} = \begin{cases} c_j^i, & \text{if } j \in \{J_i^{1<} \cup J_i^{0>}\} \\ d_j^i, & \text{otherwise.} \end{cases}$$

By contradiction, assume that there exists a solution $x \in X$ such that $D^*x$ dominates $D^*x^0$. Therefore, w.l.g., one obtains:

$$\begin{cases} \sum_{j\in J} d_j^{1*} x_j > \sum_{j\in J} d_j^{1*} x_j^0 \\ \sum_{j\in J} d_j^{i*} x_j \geqslant \sum_{j\in J} d_j^{i*} x_j^0, \text{ for all } i \in I\backslash\{1\} \end{cases}$$

One can deduce,

$$\sum_{j\in J_1^{1<}} d_j^{1*} x_j + \sum_{j\in J_1^{0>}} d_j^{1*} x_j + \sum_{j\in J_1^{0\leqslant}\cup J_1^{1\geqslant}} d_j^{1*} x_j > \sum_{j\in J_1^{1<}} d_j^{1*} x_j^0 + \sum_{j\in J_1^{1\geqslant}} d_j^{1*} x_j^0$$

$$\sum_{j\in J_1^{1<}} d_j^{1*} x_j + \sum_{j\in J_1^{0>}} d_j^{1*} x_j + \sum_{j\in J_1^{0\leqslant}\cup J_1^{1\geqslant}} d_j^{1} x_j > \sum_{j\in J_1^{1<}} d_j^{1*} x_j^0 + \sum_{j\in J_1^{1\geqslant}} d_j^{1} x_j^0$$

$$\sum_{j\in J} d_j^1 x_j - \sum_{j\in J_1^{0>}} \alpha_j^1 x_j + \sum_{j\in J_1^{1<}} \alpha_j^1 x_j > \sum_{j\in J} d_j^1 x_j^0 + \sum_{j\in J_1^{1<}} \alpha_j^1$$

where $\forall j \in J : \alpha_j^1 \in \mathbb{N}_0$. Since $\sum_{j\in J_1^{1<}} \alpha_j^1 x_j \leqslant \sum_{j\in J_1^{1<}} \alpha_j^1$,

$$\sum_{j\in J} d_j^1 x_j > \sum_{j\in J} d_j^1 x_j^0 + \underbrace{\sum_{j\in J_1^{1<}} \alpha_j^1 - \sum_{j\in J_1^{1<}} \alpha_j^1 x_j}_{\geqslant 0} + \sum_{j\in J_1^{0>}} \alpha_j^1 x_j$$

Similarily, it can be shown that, for all $i \in I\backslash\{1\}$

$$\sum_{j\in J} d_j^i x_j \geqslant \sum_{j\in J} d_j^i x_j^0$$

Therefore,

$$\begin{cases} d^{1T}x > d^{1T}x^0 \\ d^{iT}x \geqslant d^{iT}x^0, \text{ for all } i \in I\backslash\{1\} \end{cases}$$

This contradicts the feasibility of $D$ for IMKP-$\infty$, and the theorem is proved. □

Let us define a matrix $D^k \in \mathbb{N}_0^{q\times n}$ of distance at most $k$ from matrix $C$ with respect to the $L_\infty$ norm.

**Definition 3.2** ($D^k$). *Given $k \in \mathbb{N}_0$, for all $i \in I$, and $j \in J$,*

$$D_{ij}^k = \begin{cases} \max\{0, C_{ij} - k\}, & \text{if } x_j^0 = 0, \\ C_{ij} + k, & \text{otherwise.} \end{cases}$$

**Theorem 3.3.** *If $D^*$ represents an optimal solution of IMKP-$\infty$, with $k = \max_{j\in J, i\in I}\{|C_{ij} - D_{ij}|\}$, then $D^k$ is also an optimal solution of IMKP-$\infty$.*

*Proof.* Let $D^*$ denote an optimal solution of IMKP-$\infty$ with $\max_{j\in J,i\in I}\{|C_{ij} - D^*_{ij}|\} = k$, and $J^k_i = \{j \in J : (|C_{ij} - D^*_{ij}| < k) \wedge (D^*_{ij} \neq 0)\}$, for $i \in I$. From Theorem 3.1, it can be assumed that $\forall i \in I$, $\forall k \in J^1 : D^*_{ij} \geqslant C_{ij}$ and $\forall j \in J^0 : D^*_{ij} \leqslant C_{ij}$. Therefore, if $|\cup_{i\in I} J^k_i| = 0$, then $D^* = D^k$. Assume that $|\cup_{i\in I} J^k_i| \geqslant 1$.

Assume also that there exists a solution $x \in X$ such that $D^k x$ dominates $D^k x^0$. Therefore, w.l.g., one obtains:

$$\begin{cases} \sum_{j\in J^k_1} d^{1k}_j x_j + \sum_{j\in J\setminus J^k_1} d^{1k}_j x_j > \sum_{j\in J^k_1} d^{1k}_j x^0_j + \sum_{j\in J\setminus J^k_1} d^{1k}_j x^0_j \\ \sum_{j\in J^k_i} d^{ik}_j x_j + \sum_{j\in J\setminus J^k_i} d^{ik}_j x_j \geqslant \sum_{j\in J^k_i} d^{ik}_j x^0_j + \sum_{j\in J\setminus J^k_i} d^{ik}_j x^0_j, \text{ for all } i \in I \end{cases}$$

This system can be rewritten as follows:

$$\begin{cases} \sum_{j\in J^k_1} d^{1k}_j x_j + \sum_{j\in J\setminus J^k_1} d^{1*}_j x_j > \sum_{j\in J^k_1} d^{1k}_j x^0_j + \sum_{j\in J\setminus J^k_1} d^{1*}_j x^0_j \\ \sum_{j\in J^k_i} d^{ik}_j x_j + \sum_{j\in J\setminus J^k_i} d^{i*}_j x_j \geqslant \sum_{j\in J^k_i} d^{ik}_j x^0_j + \sum_{j\in J\setminus J^k_i} d^{i*}_j x^0_j, \text{ for all } i \in I \end{cases}$$

Some properties based on the cardinality of $J^k_i$ can be deduced. When the cardinality equals zero, the two following conditions are obvious:

- $|J^k_1| = 0 : \sum_{j\in J} d^{1*}_j x_j > \sum_{j\in J} d^{1*}_j x^0_j$
- $|J^k_i| = 0 : \sum_{j\in J} d^{i*}_j x_j \geqslant \sum_{j\in J} d^{i*}_j x^0_j$, for all $i \in I$

Otherwise, consider the case where the cardinality is greater or equal to one:

- $|J^k_1| \geqslant 1$ :

  $$\sum_{j\in J^k_1} d^{1k}_j x_j = \sum_{j\in J^k_1\cap J^1} (c^1_j + k)x_j + \sum_{j\in J^k_1\cap J^0} (c^1_j - k)x_j$$

  By definition of $J^k_1$,

  $$\sum_{j\in J^k_1} d^{1k}_j x_j = \sum_{j\in J^k_1} d^{1*}_j x_j - \sum_{j\in J^k_1\cap J^1} \alpha^1_j x_j + \sum_{j\in J^k_1\cap J^0} \alpha^1_j x_j$$

  where $\forall j \in J : \alpha^1_j \in \mathbb{N}_0$. Furthermore,

  $$\begin{aligned} \sum_{j\in J^k_1} d^{1k}_j x^0_j &= \sum_{j\in J^k_1\cap J^1} (c^1_j + k)x^0_j \\ &= \sum_{j\in J^k_1\cap J^1} d^{1*}_j x^0_j - \sum_{j\in J^k_1\cap J^1} \alpha^1_j \end{aligned}$$

  Finally, since $\sum_{j\in J^k_1\cap J^1} \alpha^1_j x_j \leqslant \sum_{j\in J^k_1\cap J^1} \alpha^1_j$ it can be concluded that $\sum_{j\in J} d^{1*}_j x_j > \sum_{j\in J} d^{1*}_j x^0_j$.
- $|J^k_i| \geqslant 1$ : Similarly one obtains $\sum_{j\in J} d^{i*}_j x_j \geqslant \sum_{j\in J} d^{i*}_j x^0_j$.

Therefore, with $|\bigcup_{i\in I} J^k_i| \geqslant 1$, in all the cases $D^* x$ dominates $D^* x^0$ which contradicts the feasibility of $D^*$ for IMKP-$\infty$, and the theorem is proved. $\square$

**Lemma 3.4.** *If $\delta_\infty \in \mathbb{N}_0$ is the optimal solution value of IMKP-$\infty$, then $\delta_\infty \leqslant \Delta = \max_{i\in I, j\in J}\{(1-x_j^0)c_j^i\}$*

*Proof.* It is always possible to build a matrix $D \in \mathbb{N}_0^{q\times n}$ with $\max_{i\in I,j\in J}\{|c_j^i - d_j^i|\} = \max_{i\in I,j\in J^0}\{c_j^i\}$ such that $Dx^0$ is a non-dominated solution of $(X, D)$. The matrix is defined as follows, $\forall i \in I, \forall j \in J^1 : d_j^i = c_j^i$ and $\forall i \in I, \forall j \in J^0 : d_j^i = 0$. It is easy to see that for all $x \in X$, one obtains $Dx^0 \geqq Dx$ and $\max_{i\in I,j\in J}\{|c_j^i - d_j^i|\} = \max_{i\in I,j\in J^0}\{c_j^i\}$. This concludes the proof. □

**Lemma 3.5.** *If $D^k x^0$ is a non-dominated vector for $(D^k, X)$, then $D^{k+1}x^0$ is a non dominated vector for $(D^{k+1}, X)$.*

*Proof.* Assume that there exists a solution $x \in X$ such that $D^{k+1}x$ dominates $D^{k+1}x^0$. Therefore, w.l.g., one obtains:

$$\begin{cases} \sum_{j\in J} d_j^{1,k+1}x_j > \sum_{j\in J} d_j^{1,k+1}x_j^0 \\ \sum_{j\in J} d_j^{i,k+1}x_j \geqslant \sum_{j\in J} d_j^{i,k+1}x_j^0, \text{ for all } i \in I \end{cases}$$

From the definitions of $D^k$, $J^1$, and $J^0$, one obtains:

$$\begin{cases} \sum_{j\in J^1}(d_j^{1,k}+1)x_j + \sum_{j\in J^0}(d_j^{1,k+1} - \alpha_j^1)x_j > \sum_{j\in J^1}(d_j^{1,k}+1)x_j^0 \\ \sum_{j\in J^1}(d_j^{i,k}+1)x_j + \sum_{j\in J^0}(d_j^{i,k+1} - \alpha_j^i)x_j \geqslant \sum_{j\in J^1}(d_j^{i,k}+1)x_j^0, \text{ for all } i \in I \end{cases}$$

where $\forall j \in J : \alpha_j^1, \alpha_j^2 \in \{0, 1\}$. This can be rewritten as follows:

$$\begin{cases} \sum_{j\in J} d_j^{1,k}x_j > \sum_{j\in J} d_j^{1,k}x_j^0 + \underbrace{|J^1| - \sum_{j\in J^1} x_j}_{\geqslant 0} + \sum_{j\in J^0}\alpha_j^1 x_j \\ \sum_{j\in J} d_j^{2,k}x_j \geqslant \sum_{j\in J} d_j^{2,k}x_j^0 + \underbrace{|J^1| - \sum_{j\in J^1} x_j}_{\geqslant 0} + \sum_{j\in J^0}\alpha_j^2 x_j \end{cases}$$

Therefore, $D^k x$ dominates $D^k x^0$ which contradicts the hypothesis, and the lemma is proved. □

Based on the previous results, an algorithm for computing an optimal solution of IMKP-$\infty$ is devised. Thanks to Theorem 3.3, an optimal solution of IMKP-$\infty$ can be built based on the distance between matrices $D^*$ and $C$. Since this distance is bounded from above by $\Delta = \max_{i\in I,j\in J}\{(1-x_j^0)c_j^i\}$ (see Lemma 3.4), the algorithm consists of finding the minimal value $k \in \{1, 2, \dots, \Delta\}$ such that $D^{kT}x^0$ is a non-dominated vector of the multi-objective knapsack instance $(X, D^k)$. This condition is naturally checked by solving Problem 1. The minimal value of $k$ can be found by performing a binary search on the set $\{1, 2, \dots, \Delta\}$ as a consequence of Lemma 3.5. Therefore, Algorithm 1 requires to solve $O(log_2\Delta)$ times Problem 1. For more details on the procedure, see the pseudo-code of Algorithm 1.

# 4 Conclusion

In this paper, the inverse multi-objective knapsack problem has been defined and studied for the Chebyshev distance. The algorithm requires only to solve a logarithmic number of mixed integer programs containing a linear number of constraints and variables compared to the instance of the knapsack problem.

There are many possible extensions of this work. For instance, the use of other norms, the nature of the adjustment, and the introdution of some constraints on the adjustment are of obvious interest. Moreover, since there is not only one way to solve multi-objective optimization problems one can define inverse multi-objective optimization differently leading thus to a wide range of models and applications.

**Algorithm 1** Compute an optimal solution $D^*$ of IMKP-$\infty$.

```
a ← 0;
b ← C;
while (a ≠ b) do
    k ← a + ⌊(b − a)/2⌋;
    for all (j = 1 to n) do
        if (x_j^0 = 0) then
            for all (i = 1 to q) do
                D_ij^k ← max{0, C_ij − k};
            end for
        else if (x_j^0 = 1) then
            for all (i = 1 to q) do
                D_ij^k ← C_ij + k;
            end for
        end if
    end for
    s* ← Solve Problem 1;
    if (∑_{i∈I} s_i* = 0) then
        b ← k;
    else
        a ← k + 1;
    end if
end while
D* ← D^k;
```

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